

Langevin dynamics of financial systems: A second-order analysis

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Abstract. We address the issue of stock market fluctuations within Langevin Dynamics (LD) and the thermodynamics definitions of multifractality in order to study its second-order characterization given by the analogous specific heat C_q , where q is an analogous temperature relating the moments of the generating partition function for the financial data signals. Due to non-linear and additive noise terms within the LD, we found that C_q can display a shoulder to the right of its main peak as also found in the S&P500 historical data which may resemble a classical phase transition at a critical point.

PACS. 02.50.Ey Stochastic processes – 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 89.90.+n Other topics in areas of applied and interdisciplinary physics

1 Introduction

Simulated descriptions of at least some universal features of financial prices has recently been produced by multifractal techniques [1] and also linear [2,3] and non-linear [4] Langevin Dynamics (LD). On one hand, multifractals reasonably describe patterns that resemble sudden (both large and small) market price oscillations, ensuing bursts of long-range volatility as in the case of stochastic multi-agent models [5]. On the other hand, non-linear [4] LD generates fat tail distributions of price differences as found in the analysis of financial time series [6,7].

There is a vast literature on the characterization of financial data based on turbulence, multifractality, auto-correlation functions and power spectra studies (see, *e.g.*, [8–12]). However the second-order derivative relating the increment of price values of assets $\Delta x(t, 1) \equiv x(t+1) - x(t)$ and given by an analogous specific heat C_q similarly to the multifractal studies of diffusion-limited aggregates (DLA) [13], has only recently been analysed by the author in the case of temporal fluctuations of the S&P500 stock index [14]. The analogous specific heat is given by the second order derivative of the analogous free energy τ relating the multifractal dimension D_q (*cf.*, Eqs. (6) and (7) below). The q -moments of the generating partition function for the financial data signals $\Delta x(t, 1)$ corresponds to an analogous temperature.

As an illustrative example, the curve (a) in the lower Figure 1 shows the presence of an anomalous shoulder in C_q when considering 3287 (S&P500) data points including the largest burst that corresponds to the so-called *Black Monday* crash measured in October of 1987. For

comparison, in the plotted curves (b, c, d) of Figure 1 we remove the crash from the historical data and make the measurement of C_q anew. Curve (b) is the result of just removing the single data point for the *Black Monday* (*i.e.*, by considering 3286 points), whereas the curve (c) is the result of removing the *Black Monday* plus all successive data (*i.e.*, considering 1970 initial points). On the other hand, curve (d) is the result of taking 499 initial data points only.

Clearly, as one removes or approaches the *Black Monday* data the shoulder in C_q of curve (a) does not disappear as can be seen in curves (b, c). A similar behaviour is observed analysing the data from few days prior to the S&P500 crash. As a difference C_q only displays a typical, single peaked form without anomalies for the data prior to, but far away from, the *Black Monday* (as in curve d).

In this work we make an attempt to characterize the onset of market crashes by analysing the second-order derivative relating the increment of prices of assets $\Delta x(t, 1)$ and given by C_q . We shall consider the LD of financial systems [3,4] to show that the presence of an anomalous shoulder in the specific heats as those plotted in Figure 1 can also be a consequence of having non-linear stochastic terms within LD.

2 Linear and non-linear Langevin dynamics

We consider the differential equation with an additive noise term of the form g_t times η_t :

$$\frac{dx(t)}{dt} = h_t(x(t)) + g_t(x(t))\eta_t, \quad (1)$$

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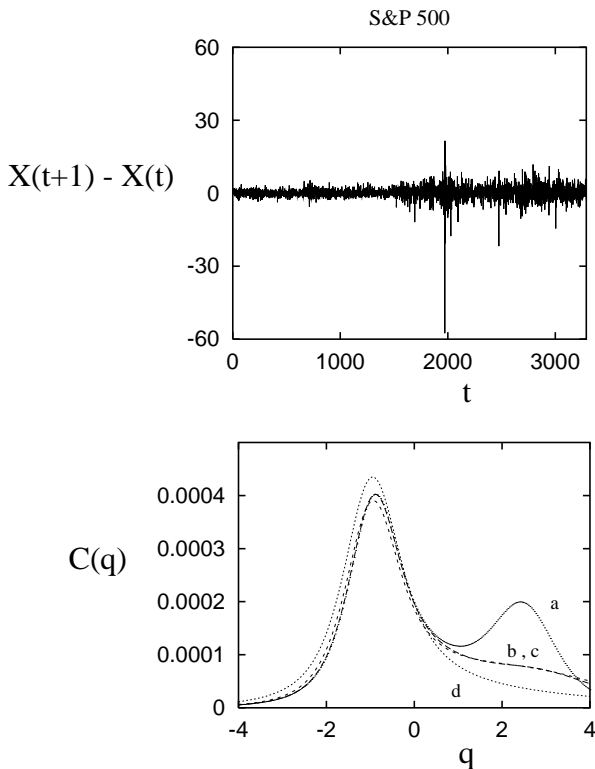


Fig. 1. Time evolution $\Delta x(t, 1)$ of the S&P500 stock index for the period 1980-1992 as a function of trading time lags (in *a.u.*) and corresponding analogous specific heat C_q obtained using equation (7). The lower curve (a) shows the presence of an anomalous shoulder in C_q when considering 3287 (S&P500) data points including the largest burst that corresponds to the crash measured in October of 1987. Curve (b) is the result of removing this *Black Monday* crash data point, whereas curve (c) is the result of removing all successive data. Curve (d) is the result of taking 499 initial data points only.

where as usual η_t is assumed to be Gaussian-white whose average and variance are given by $\overline{\eta_t} = 0$ and $\overline{\eta_t \eta_{t'}} = 2D\delta(t - t')$, respectively.

The dynamics of simplest financial systems with random noise has been proposed in [3, 15, 16] by setting

$$h_t(x(t)) \equiv -\lambda_t x(t); \quad g_t(x(t)) \equiv 1, \quad (2)$$

with λ_t a multiplicative noise that sets the time scale for deviations from equilibrium. In the context of economic models, $x(t)$ may represent price values, λ_t the rate of change of the price, and η_t some external noise of various sources. In the following we shall refer equation (2) as the linear LD. We add that the scaling behaviour and the $1/f$ noise of trajectories and velocities with time within linear LD with fluctuating random forces has been reported in our previous work [17].

Another simple financial agent model has been introduced in [4] in analogy with the mean field approach to the Ising model for a magnetic system characterized by

$$h_t(x(t)) \equiv Jx(t) + bx^2(t) - cx^3(t); \quad g_t(x(t)) \equiv e + x(t), \quad (3)$$

with J , b and c constants associated with higher-order agent interactions leading to power laws tails. The parameter e allows this agent model to take up the simple Langevin form when $x(t) \rightarrow 0$. The dynamics of time series from stochastic processes governed by a similar non-linear LD has also been reported in [18].

3 Multifractality and analogous specific heat

In accordance with the standard of economic notation only relative changes $\Delta x(t, 1)$ are usually relevant. From these fluctuations, we follow [14] and construct the generating partition function Z in terms of the normalized measures $\mu_t > 0$, and its moments q , by the scaling

$$Z(q, N) = \sum_{t=1}^N \mu_t^q \sim N^{-\tau(q)}, \quad (4)$$

where

$$\mu_t \equiv \frac{|\Delta x(t, 1)|}{\sum_{t=1}^N |\Delta x(t, 1)|}. \quad (5)$$

In the above we have divided the 1D system of length L into N lines of length ℓ (*i.e.*, $N \sim L/\ell$), and have associated this N with the measured discrete $x(t)$ sequences [19].

In order to get the thermodynamics interpretation of multifractality (see, *e.g.* [13]), we then consider standard definitions

$$\tau(q) \equiv [q - 1]D_q, \quad (6)$$

where τ represents an analogous free energy and D_q the multifractal dimension. Of particular interest here is to consider the analogous specific heat given by the second order derivative of τ , namely

$$-C_q \equiv \frac{\partial^2 \tau(q)}{\partial q^2} \approx \tau(q+1) - 2\tau(q) + \tau(q-1). \quad (7)$$

4 Discussion

In the upper curves of Figure 2 we show some typical time evolution of $\Delta x(t, 1)$ governed by the linear LD with h_t and g_t given in equation (2) for different values of $\eta_t = (1 - 2r_{n'})v$ and $\lambda_t = (1 - 2r_n)u$ with $v = 0.1$, $u = -0.005, -0.02$ and -0.1 , and using the random numbers $0 \leq r_{n, n'} \leq 1$. In the lower curves of Figure 2 we display the non-linear LD fluctuations of $\Delta x(t, 1)$ with h_t and g_t given in equation (3) with $J = -1$, $b = 0.01$, $c = 0.001$, $e = 0.005$ and for different values of $\eta_t = (1 - 2r_{n'})n$ such that $n = 1.2, 1.5$ and 1.8 and using random numbers $0 \leq r_{n'} \leq 1$.

From the curves in Figure 2 it can be seen that $\Delta x(t, 1)$ has relatively small finite values, and exhibits intermittent bursts due to the weak noise terms. The presence of peaked narrow fluctuations can increase with both the

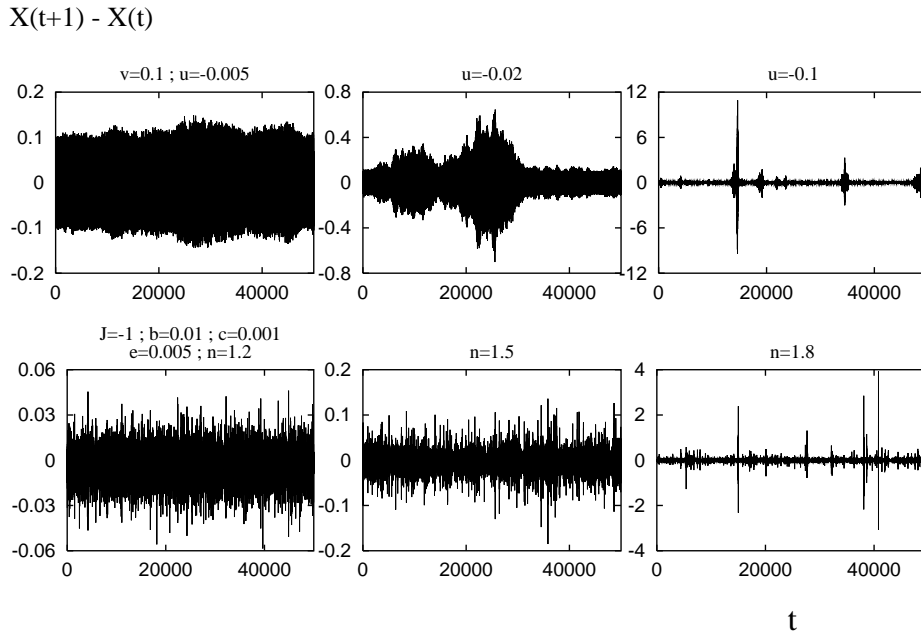


Fig. 2. Time evolution of $\Delta x(t, 1)$ governed by the LD of equation (1) using the linear terms in equation (2) (upper curves) and non-linear terms of equation (3) (lower curves).

multiplicative and additive noise strength. However, wider volatility clusters are formed over large periods of time when decreasing, in the linear LD, the amplitude u of the multiplicative λ_t noise as compared to the case of increasing n of the additive noise η_t within the non-linear LD. It is this difference of behaviour that will lead C_q to display an anomalous shoulder to the right of its main peak to be shown below.

Let us first see if the generalized dimensions D_q for both linear and non-linear LD, which are extracted by considering the definition in equation (6) in conjunction with the scaling of equation (4), are sufficiently smooth and multifractal for C_q of equation (7) to be meaningful [20, 21]. The discrete values of q used range from -5 to $+5$ at increments of 0.033 .

In Figure 3 we display the plots of D_q for the different values of u and n as used in Figure 2 together with the plots of the limit $1 - D_{q \rightarrow \infty}$ for the linear and non-linear LD (upper and lower curves, respectively). If $q < -1$, we observe that both results for D_q follow a typical convergent behaviour according to multifractal physics. In the range $u > -0.02$ and $n < 1.5$, D_q is fully multifractal-like.

For lower values of $u < -0.02$ and greater values of $n > 1.5$, we find a non-monotonous decreasing behaviour of D_q , in correspondence with the double-peaked form of the respective C_q displayed in Figure 4 which relates the presence of the intermittent bursts in $\Delta x(t, 1)$ shown in Figure 2. Such a non-monotonous behaviour of D_q is more evident in the case of non-linear LD. From the D_q data for the linear LD case, the multifractality strength of the $\Delta x(t, T) \equiv x(t+T) - x(t)$ sequences at different integer time lags $T \geq 1$, *i.e.*, $1 - D_{q \rightarrow \infty}$, does not seem to follow a power-law scaling for $1 < T < 10$ within all u considered. As a consequence of the complicated behaviour of the

non-linear form of equation (3) coupled to η_t , which influences the (thinner) volatility clusters shown in Figure 2 (lower curves), it is not straightforward that $1 - D_{q \rightarrow \infty}$ in this case follows a power scaling as in [21].

Let us see next how this type of D_q behaviour given in Figure 3 influences the analogous specific heat of equation (7) and how it characterizes the temporal large and small intermittent bursts representing variations in financial prices. In Figure 4 we plot C_q for the LD parameter set used in Figures 2 and 3. For the linear LD case, we find that the main peak of our numerical C_q curves for $u > -0.02$ resembles a classical first-order phase transition at a critical point $q = -1$. However, the sharp peak turns slightly asymmetric around the critical point for $u = -0.1$. Surprisingly, this asymmetry becomes evident within the non-linear LD. The analogous specific heat in this case displays a higher shoulder to the right of the main peak as a function of $n \geq 1.5$ which is in qualitative agreement with the C_q curve for the S&P500 index data shown in Figure 1 (lower curves). The presence of the shoulder in C_q is then associated to the non-monotonous behaviour of D_q as in Figure 3.

The same C_q shape at a phase transition as in Curve (d) of Figure 1 and Figure 4, has been reported in the case of DLA [13]. A non monotonic specific heat shape with a superimposed second peak (resembling the shoulder of the type of Curve (a) in Fig. 1 and Fig. 4 lower curves) has been found in the Hubbard model on a cluster of magnetic sites [22, 23] and, most recently, in a uniform spin model on a fractal [24].

When decreasing $n \leq 1.2$ in the non-linear LD, the sudden intermittent bursts in $\Delta x(t, 1)$ as well as the shoulder in C_q tend to vanish. It is this latter feature that make us argue that the stochastic price behaviour of financial

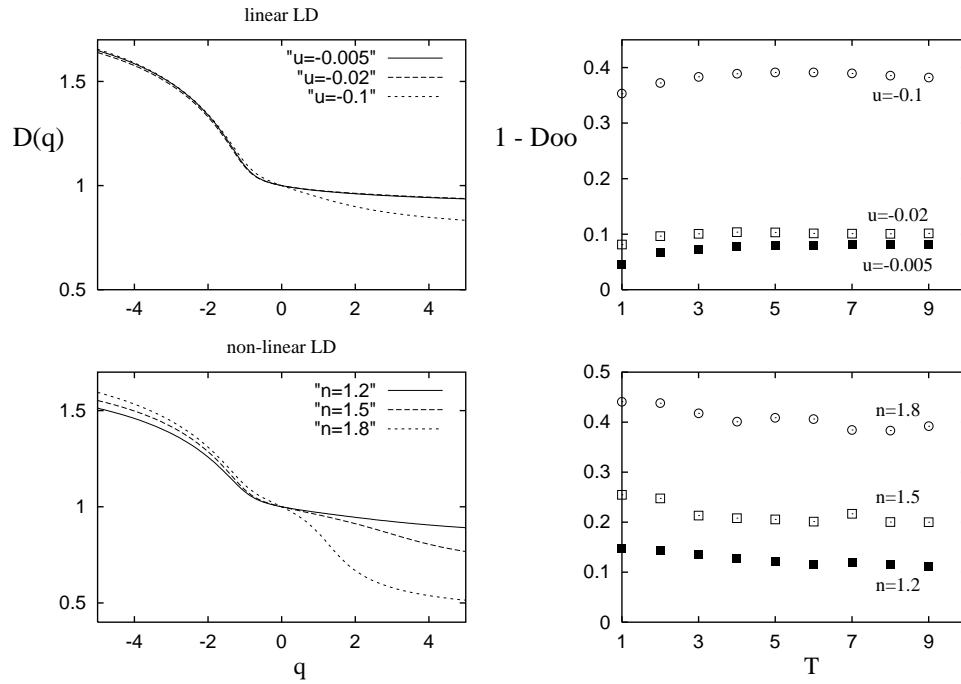


Fig. 3. Generalized dimensions D_q and multifractality strength $1 - D_{q \rightarrow \infty}$ for different time lags T for the linear LD using equation (2) (upper curves) and the non-linear LD using equation (3) (lower curves). The $1 - D_{q \rightarrow \infty}$ curves have been evaluated by considering $\Delta x(t, T) \equiv x(t + T) - x(t)$.

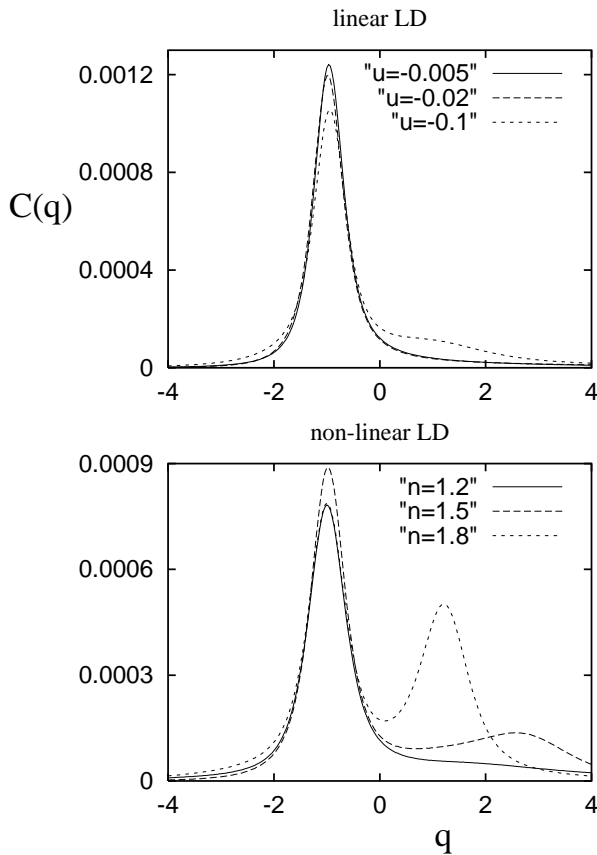


Fig. 4. Analogous specific heat C_q of the linear (upper curve) and non-linear (lower curve) LD derived using equation (7).

assets may be characterized by an analogous C_q which resembles the phase transition features as measured in multifractal physics.

5 Conclusion

We have made a novel attempt to characterize the onset of market crashes *via* an analogous specific heat using the thermodynamics definitions of multifractality. We presented an study of generalized dimensions D_q of equation (6), and the analogous specific heat C_q of equation (7), characterizing the absolute moments of the increments of two LD, a linear one (*cf.*, Eq. (2)) and a non-linear one (*cf.*, Eq. (3)), within the context of a multifractal approach to financial time series analysis [14].

Tuning parameters in these dynamics introduces wide clusters and bursts of volatility in the processes. This behaviour was shown to mimics characteristics of S&P500 data which were measured according to an analogous phase transition in the behaviour of C_q .

In this work, the second-order derivative relating the increment of prices of assets $\Delta x(t, 1)$, and given by C_q , follows similar Legendre transforms for the thermodynamics analysis of (DLA) [13]. Our findings may relate the results of the kind of those described in [25] where a phase transition-like appears in the presence of two distinct phases when analysing the behaviour of D_q . In the latter, these two phases are built out of rare events on the one hand and some “background” (*e.g.*, bursts of volatility in the processes) on the other hand. We have found that a shoulder appears in the analogous C_q due to the

presence of wide volatility clusters as found in the S&P500 historical data, or in the non-linear LD.

The message here is that due to non-linear and additive noise terms within the LD, C_q can display a shoulder to the right of its main peak data as also found in the S&P500 historical data, which may resemble a classical phase transition at a critical point.

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